MINIMAL IMMERSIONS OF CLOSED SURFACES IN HYPERBOLIC THREE-MANIFOLDS

ZHENG HUANG AND MARCELLO LUCIA

ABSTRACT. We study minimal immersions of closed surfaces (of genus $g \geq 2$) in hyperbolic three-manifolds, with prescribed data $(\sigma, t\alpha)$, where σ is a conformal structure on a topological surface S, and αdz^2 is a holomorphic quadratic differential on the marked Riemann surface (S, σ). We show that, for each $t \in (0, \tau_0)$ for some $\tau_0 > 0$, depending only on (σ, α), there are at least two minimal immersions of closed surface of prescribed second fundamental form $Re(t\alpha)$ in the conformal structure σ. Moreover, for t sufficiently large, there exists no such minimal immersion. Asymptotically, as $t \to 0$, the principal curvatures of one minimal immersion tend to zero, while the intrinsic curvatures of the other blow up in magnitude.

1. INTRODUCTION

A fundamental problem in hyperbolic geometry is the interaction between the hyperbolic structures of closed surfaces and those of three-manifolds. Minimal surface theory has been intimately related to the geometry and topology of three-manifolds (see for instance [**SY79, MIY82**]). It follows from [**FHS83**] that any incompressible surface can be isotoped to a minimal surface in a closed Riemannian three-manifold. Hence one expects any hyperbolic three-manifold to be obtained by gluing pieces of the type $S \times (-a, a)$, for some $a > 0$, with S minimal.

Minimal surfaces play important roles in understanding the structures of 3-manifolds (see for example [**SU82**] and recent series of work by Colding-Minicozzi [**CM04a, CM04b, CM04c, CM04d**]). Closed surfaces (compact without boundary) cannot be minimally embedded in $\mathbb{R}^3$. In the positive curvature case, the situation is quite different, since Lawson proved in [**LJ70**] that every compact orientable surface can be minimally immersed in the sphere S^3. The case of minimal immersions in hyperbolic three-manifolds is much more subtle, and has been studied by several authors (see for example [**Tau04, Rub05, Has05**]). In particular, Uhlenbeck in [**Uhl83**] has undertaken a program to parametrize a class of hyperbolic three-manifolds by incompressible minimal surfaces.

The goal of this paper is to investigate closed minimal surfaces of genus at least two immersed in hyperbolic three-manifolds, and to prove several results inspired by Uhlenbeck's approach. These surfaces admit hyperbolic metrics via the uniformization theorem, and the conformal change function between the unique hyperbolic

metric on a marked surface (S, σ) and the induced metric from the immersion into a hyperbolic three-manifold satisfies the Gauss equation, which is a semilinear elliptic equation, and we study the solution curve to a family of Gauss equations and their geometrical implications.

Throughout the paper, we assume S as a closed oriented surface of genus $g \geq 2$. Teichmüller space of S is denoted by $T_g(S)$, and it is the space of conformal structures (or equivalently hyperbolic metrics) on S such that two conformal structures σ and ρ are equivalent if there is an orientation-preserving diffeomorphism in the homotopy class of the identity between them. Let (S, σ) be the surface S marked with the conformal structure $\sigma \in T_g(S)$, and $z = x + iy$ be the conformal coordinates on (S, σ), we denote the unique hyperbolic metric on (S, σ) by $g_\sigma dz d\bar{z}$. When (S, σ) is immersed in some hyperbolic three-manifold M, its induced metric from the immersion by $f(z)dz d\bar{z}$ with $h = h_{11}dx^2 + 2h_{12}dxdxy + h_{22}dy^2$ the second fundamental form. Then it is well-known that ([**Hop89, LJ70**]) the form $\alpha = (h_{11} - ih_{12})dz^2$ is a holomorphic quadratic differential and $h = Re(\alpha)$.

From the prescribed data $(\sigma, t\alpha)$, our goal is to construct hyperbolic three-manifolds such that the closed surface S is minimally immersed. In this aspect, our main result should be considered as a "local realization theorem". This minimal immersion is governed by six equations: Three of them are in the form of curvature relation since we require the normal bundle as a three-manifold is hyperbolic: $R_{i3j3} = -g_{ij}$ (see (2.1)). They can be reduced to a system of ODEs and they determine the metric explicitly in the normal bundle of the minimal surface S (see §2.2); Two more relations are provided by the Codazzi equations: $R_{ijk3} = 0$, and they ensure the requirement for the prescribed second fundamental form (also §2.2). The last equation is the Gauss equation which ensures the minimal immersions stay in the prescribed conformal structure σ. Using these equations, we build a hyperbolic three-manifold, topologically $S \times (-a, a)$, for $a \in (0, \infty]$, around the minimal surface S. Among these equations, probably the Gauss equation is the most intriguing. The conformal factor between the induced metric $f(z)dz d\bar{z}$ and the hyperbolic metric $g_\sigma dz d\bar{z}$ on (S, σ) can be represented via $f(z) = e^{2u(z)}g_\sigma(z)$, and the Gauss equation is an elliptic semilinear equation given by

$$(1.1) \qquad \Delta u + 1 - e^{2u} - \frac{|\alpha|^2}{g_\sigma^2}e^{-2u} = 0,$$

where Δ is the Laplacian in the hyperbolic metric $g_\sigma dz d\bar{z}$.

Definition 1.1. *We call $S(\sigma, \alpha)$ a minimal immersion with data (σ, α) if S is marked by a conformal structure $\sigma \in T_g(S)$ and S is a minimal immersion whose second fundamental form is given by $Re(\alpha)$, for $\alpha \in Q(\sigma)$.*

We consider a ray $t\alpha(z)dz^2$, for a fixed direction $\alpha \in Q(\sigma)$, and $t \geq 0$. Note that the space of holomorphic quadratic differentials, $Q(\sigma)$, on (S, σ) is identified as the cotangent space of Teichmüller space at the point $\sigma \in T_g$, therefore $t\alpha$ represents a ray in $Q(\sigma)$, and this ray is closely related to the notion of Teichmüller geodesics in Teichmüller space. The data (σ, α) is a point in the cotangent bundle $T_g(S) \times Q(\sigma)$, where $Q(\sigma)$ is a Banach space of real dimension $6g - 6$. This ray enables us to study

the one-parameter family of Gauss equations:

$$(1.2) \qquad \Delta u(t) + 1 - e^{2u(t)} - \frac{t^2 |\alpha|^2}{g_\sigma^2} e^{-2u(t)} = 0,$$

for minimal immersions $S(\sigma, t\alpha)$.

Using the implicit function theorem, Uhlenbeck ([**Uhl83**]) proved the existence of a smooth solution curve to the equation (1.2):

Theorem [Uhl83]. *Fixing a conformal structure $\sigma \in T_g(S)$, and $\alpha \in Q(\sigma)$, there exists a constant $\tau_0 > 0$, depending only on (σ, α), such that for each $t \in [0, \tau_0]$, there is a stable minimal immersion of S with data $(\sigma, t\alpha)$ into some hyperbolic three-manifold.*

Our main result in this paper is to obtain an additional solution for each Uhlenbeck's nonzero stable solution to the Gauss equation in this paper, which can be formulated as the following theorem:

Theorem 1.2. *Let S be a closed surface and $\sigma \in T_g(S)$ be a conformal structure on S. If $\alpha \in Q(\sigma)$ is a holomorphic quadratic differential on (S, σ), then:*
  (i) *for sufficiently large t, the Gauss equation (1.2) admits no solutions, i.e., there is no minimal immersion of S with data $(\sigma, t\alpha)$ into some hyperbolic three-manifolds;*
  (ii) *there exists a constant $\tau_0 > 0$, such that, for each $t \in (0, \tau_0)$, there exist at least two minimal immersions of S into some hyperbolic three-manifold in the conformal class of σ with the second fundamental form $Re(t\alpha)$.*

In Uhlenbeck's theorem, she also proved that there is a positive constant ϵ, such that there is an unstable solution on for each $t \in (\tau_0 - \epsilon, \tau_0)$. We point out that in this parameter interval $(\tau_0 - \epsilon, \tau_0)$, the minimal immersion obtained from our additional solution might coincide with Uhlenbeck's unstable solution on the same interval.

The nature of these solutions indicates important geometric information on the minimal surfaces, as well as the hyperbolic three-manifolds they immerse into: At $t = 0$, the surface S is totally geodesic, and its normal bundle is a Fuchsian manifold, i.e., a warped product hyperbolic three-manifold. For t small enough along the Uhlenbeck solution curve, and the principal curvatures of the minimal immersion stay bounded in magnitude less than 1, its normal bundle is a so-called *almost Fuchsian manifold*, and S is the unique minimal surface within this normal bundle ([**Uhl83**]). Then further along the solution curve, the minimal immersions remain stable until a particular parameter value, but the normal bundle becomes finite. It will be very interesting to understand further solutions along this solution curve, as well as the geometry of hyperbolic three-manifolds when t is approaching its maximal value when such a minimal immersion is allowed.

In Theorem 1.2, we use Uhlenbeck's parameterization of the solution curve to study the equation (1.2), and find an additional solution for each Uhlenbeck's stable solution along the solution curve. We note that though these solutions represent the same point in Teichmüller space, the hyperbolic three-manifolds they immersed into

are quite different: by fixing the data $(\sigma, t\alpha)$, we fix the conformal class of the surface and its second fundamental form for the minimal immersion. Different solutions represent different induced metrics on the surface (S, σ), hence two normal bundles are distinct. It is an intriguing question to ask how many minimal immersions are allowed for each given data $(\sigma, t\alpha)$.

Naturally we also consider the asymptotic behavior of the solutions of the Gauss equation (1.2). The blow-up analysis near $t = 0$ provides important structural information about the minimal immersions we obtained in Theorem 1.2:

Theorem 1.3. *Let $\{u_n(t_n)\}$ be a sequence of solutions to the equation* (1.2) *with $t_n \to 0$. Then, along a subsequence, the following alternative holds*

(i) *u_n coincides with the solution obtained in Uhlenbeck's theorem, in which case, the normal bundle of S is an almost Fuchsian three-manifold, and the principal curvatures are less than one in absolute value;*

(ii) *or $\|u_n\|_\infty \to \infty$, in which case, the absolute values of the intrinsic curvatures of the corresponding minimal immersion go to infinity.*

Our technique is to study the variational theory for the solutions to the Gauss equation. In calculus of variations, the problem of obtaining additional solutions to some differential equation is well-studied ([**GT83**], [**Str00**]): one rewrites the equation such that the nonlinear operator is the derivative of an appropriate functional, and uses techniques such as the mountain pass theorem from nonlinear functional analysis to find other critical points of the associated functional. Much of the difficulty is that the usual variational setting for the problem does not satisfy the compactness property. We introduce a different but equivalent inner product structure to the usual Sobolev space for the problem and prove the mountain pass theorem in the new setting.

Plan of the paper. We will collect preliminary results in section two. In particular, we briefly introduce hyperbolic geometry of dimensions two and three, and set up the Gauss equation in this setting. Section three is devoted to prove our main results, and it breaks into several subsections: in §3.1, we prove a nonexistence theorem for large parameter t; in §3.2, we study Uhlenbeck's solution curve and its parameterization; in §3.3, we work in the variational setting of the problem, and define a new norm and show that the functional with the norm satisfies the Palais-Smale compactness condition in §3.4, therefore develop the mountain pass structure of the solutions; and in the section §3.5, we prove the Theorem 1.3.

2. Preliminaries

2.1. **Hyperbolic surfaces and three-manifolds.** By hyperbolic spaces, we refer to Riemannian manifolds of constant sectional curvature -1. Naturally, a hyperbolic manifold M^n of dimension n is a quotient space of $\mathbb{H}^n$ $(n \geq 2)$, by a subgroup

of the (orientation preserving) isometry group $\text{Iso}(\mathbb{H}^n)$. We only consider $n = 2$ and $n = 3$ in this paper. These two cases are drastically different, largely due to Mostow's rigidity theorem.

In the case of $n = 2$, we have $\text{Iso}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$, which can be identified to a subgroup of $\text{Iso}(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$. Let S be a closed surface of genus $g \geq 2$, then there is a hyperbolic metric in each conformal structure of S, by the uniformization theorem. Let σ be a conformal structure on S, it is a point in Teichmüller space $T_g(S)$. We often use z and $g_\sigma dz d\bar{z}$ to record the conformal coordinate and the hyperbolic metric on (S, σ). Similarly, we use w and $g_\rho dw d\bar{w}$ to record the conformal coordinate and the hyperbolic metric on another conformal structure $\rho \in T_g(S)$. The geometry of Teichmüller space is often studied via its cotangent bundle. At $\sigma \in T_g(S)$, we have the cotangent space $Q(\sigma)$, where $\alpha \in Q(\sigma)$ is a holomorphic quadratic differential on (S, σ). Locally, $\alpha = \alpha(z) dz^2$, where $\alpha(z)$ is holomorphic.

Let $M^3 = \mathbb{H}^3/\Gamma$ be a hyperbolic three-manifold, and we assume $\Gamma \subset \text{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ properly and discontinuously. In this case we call Γ a *Kleinian group*. For any $p \in \mathbb{H}^3$, the orbit set of Γ has accumulation points on the boundary S^2_∞. The closed set of these limit points is called the *limit set* Λ_Γ of the group Γ. There are two elementary types of hyperbolic three-manifolds we will encounter frequently: when Λ_Γ is a round circle, M^3 is called *Fuchsian*, which is a product space of a hyperbolic surface S and the real line $\mathbb{R}$. It is easy to see that the space of Fuchsian manifolds is isometric to Teichmüller space; when Λ_Γ lies in a Jordan curve, M^3 is called *quasi-Fuchsian*, and it is topologically $S \times \mathbb{R}$. In this case M^3 is a complete hyperbolic three-manifold quasi-isometric to a Fuchsian manifold. These two types correspond to the beginning of the solution curve for the Gauss equation (1.2).

In the case of M^3 being quasi-Fuchsian, it admits at least one immersed area-minimizing incompressible surface ([**SY79, SU82**]). Furthermore, if the principal curvatures are less than one in magnitude, i.e., the case of almost Fuchsian, the incompressible minimal surface is unique. Hence one can use minimal surfaces to parametrize the space of almost Fuchsian manifolds within the quasi-Fuchsian space ([**Uhl83, Tau04**]), and obtain important geometric and dynamical information about the almost Fuchsian manifolds, in terms of the geometry of the minimal surface: for example, hyperbolic volume of the convex core and the Hausdorff dimension of the limit set ([**HW09**]), and Teichmüller distance between conformal infinities ([**GHW10**]). There are also recent important work on the applications of almost Fuchsian manifolds to mathematical physics (see for instance [**KS07, KS08**]).

2.2. **The normal bundle.** Let $S \subset M^3$ be a minimal immersion of S into a hyperbolic three-manifold M^3, and $T^\perp S$ be its normal bundle in M^3. The Riemann curvature tensor $R_{ijk\ell}$ on M^3 has six components, three of them satisfy curvature equation of the form:

$$(2.1) \qquad\qquad R_{i3j3} = -g_{ij}, \quad i = (1,2), j = (1,2).$$

From classical Riemannian geometry, the exponential map $\exp : T^\perp S \to M^3$ is a local diffeomorphism on $S \times (-\epsilon, \epsilon) \subset T^\perp S$. Therefore, given first and second

fundamental forms on S, these equations (2.1) uniquely determine a hyperbolic metric on $S \times (-\epsilon, \epsilon) \subset T^\perp S$. On the normal bundle $T^\perp S$, these three equations can be reduced to a second order system for fixed $z \in S$:

$$\frac{1}{2}\frac{\partial^2 g_{ij}(z,r)}{\partial r^2} - \frac{1}{4}\frac{\partial g_{i\ell}(z,r)}{\partial r}g^{\ell k}\frac{\partial g_{kj}(z,r)}{\partial r} = g_{ij}(z,r),$$

whose solution can be written explicitly as follows ([**Uhl83**]): for $(z,r) \in S \times (-\epsilon, \epsilon)$,

$$(2.2) \qquad\qquad g(z,r) = e^{2v(z)}[\cosh(r)\mathbb{I} + \sinh(r)e^{-2v(z)}A(z)]^2 ,$$

and the hyperbolic metric on $T^\perp S$ is given as $ds^2 = g(z,r)|dz|^2 + dr^2$. Here the induced metric on S is given by $g_{ij}(z) = g_{ij}(z,0) = e^{2v(z)}\delta_{ij}$, where $v(z)$ is a smooth function on S, and the second fundamental form $A(z) = [h_{ij}]_{2\times2}$, where $z = x + \sqrt{-1}y$ is the conformal coordinates on marked surface (S, σ). With respect to these coordinates, the second fundamental form of $S \subset M^3$ can be written as

$$(2.3) \qquad\qquad h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2.$$

The remaining three curvature equations are constraint equations for the first and second fundamental form on S. Note that the minimal surface S has zero mean curvature, so we denote $\pm\lambda(z)$ the eigenvalues of $A(z)$, where $\lambda(z) \geq 0$. They are the principal curvatures of S. In this case, $h_{11} = -h_{22}$ and two of the remaining three curvature equations are the Codazzi equations: $R_{ijk3} = 0$. That is equivalent to say (2.3) becomes ([**LJ70**]):

$$(2.4) \qquad\qquad h = Re(\alpha),$$

for some $\alpha \in Q(\sigma)$, a holomorphic quadratic differential on the marked Riemann surface (S, σ). Note that the holomorphic quadratic differential α must have zeros somewhere, or $|\alpha|dzd\bar{z}$ defines a smooth flat metric on S, violating the Gauss-Bonnet theorem.

The metric $g(z,r)$ might be singular if there are conjugate points of the exponential map. It is easy to verify that when $\lambda(z) < 1$ for all $z \in S$, then the map exp has no conjugate point and the normal bundle $T^\perp S$ extends to both infinities to become a complete hyperbolic three-manifold and S is the only minimal surface (also embedded) in $T^\perp S$.

2.3. **The Gauss equation.** Five of six curvature equations for the minimal immersion $S \subset M^3$ take the form of (2.1) and (2.4). They determine the metric in the normal bundle of S and the second fundamental form of S. The sixth equation is the Gauss equation which describes the interaction between the hyperbolic structure on S and the ambient hyperbolic structure of M^3. Note that ours is slightly different from the equation in [Theorem 4.2 [**Uhl83**]] because of an obvious typo there. We recall from (1.1).

$$\Delta u + 1 - e^{2u} - \frac{|\alpha|^2}{g_\sigma^2}e^{-2u} = 0,$$

here $g_\sigma|dz|^2$ is the hyperbolic metric on (S, σ), and the conformal factor between the induced metric $f(z)|dz|^2$ and $g_\sigma|dz|^2$ is determined by $f(z) = e^{2u(z)}g_\sigma(z)$. We also use Δ to denote the hyperbolic Laplace operator on (S, σ). This equation is

similar to the prescribed scalar curvature equation studied by Kazdan-Warner in [**KW74, KW75**].

The Gauss equation is a consequence of two equivalent ways of describing the intrinsic curvature $K(z)$ induced by the minimal immersion:

$$K(z) = e^{-2u(z)}(-\Delta u(z) - 1) = -1 - \lambda^2(z),$$

where the positive principal curvature $\lambda(z)$ is given by

$$(2.5) \qquad \lambda(z) = \frac{|\alpha|}{g_\sigma} e^{-2u},$$

because of the equation (2.4).

We are particularly interested in a family of Gauss equations, corresponding to a ray $\alpha(t) = t\alpha \in Q(\sigma)$, as in (1.2):

$$\Delta u(t) + 1 - e^{2u(t)} - \frac{t^2|\alpha|^2}{g_\sigma^2} e^{-2u(t)} = 0.$$

3. Proof of main theorems

Our main theorems concern the solution curve for the family of Gauss equations (1.2) and the geometry of the minimal surfaces corresponding to these solutions. For the parameter $t \geq 0$, we study the large values first, where we prove solutions do not exist in §3.1. In the remaining sections, we focus on the range where solutions do exist, especially Uhlenbeck's stable solutions, in §3.2. We then construct the mountain pass solutions and study their asymptotic geometry in the remainder of the section.

3.1. **Non-existence result.** Let us first emphasize that equation (1.2) does not admit any solution for large value of t. More specifically, the following theorem reveals some necessary properties for solutions to the Gauss equation. In particular, it proves part (i) of the Theorem 1.2.

Theorem 3.1. (a) *Any solution u to the Gauss equation satisfies $u \leq 0$.*

(b) *For $t \geq 2\pi(2g-2)\left(\int_{S_\sigma} \frac{|\alpha|}{|g_\sigma|}\right)^{-1}$, Problem (1.2) admits no solution.*

Proof. **(a)** This is the consequence of the maximum principle: At a maximum point x_0 of a solution u, apply the maximum principle to the equation (1.1) to obtain that $0 \leq 1 - e^{2u(x_0)}$. Hence $u \leq 0$, the conclusion follows.

(b) Let (t, u) be a solution to (1.2), and $dA_\sigma = g_\sigma dz d\bar{z}$ be the hyperbolic area element on (S, g_σ). On the one hand, by integrating equation (1.2) on (S, g_σ), and using that the area of S is $2\pi(2g-2)$, we obtain

$$
\begin{aligned}
2\pi(2g-2) &= \int_S e^{2u} dA_\sigma + t^2 \int_S \frac{|\alpha|^2}{g_\sigma^2} e^{-2u} dA_\sigma \\
&> t^2 \int_S \frac{|\alpha|^2}{g_\sigma^2} e^{-2u} dA_\sigma.
\end{aligned}
\tag{3.1}
$$

On the other hand, Cauchy-Schwarz inequality and the fact that $u \leq 0$ give

$$
\begin{aligned}
\left(\int_S \frac{|\alpha|}{g_\sigma} dA_\sigma \right)^2 &= \left(\int_S e^u \frac{|\alpha|}{g_\sigma} e^{-u} dA_\sigma \right)^2 \\
&\leq \left(\int_S e^{2u} dA_\sigma \right) \left(\int_S \frac{|\alpha|^2}{g_\sigma^2} e^{-2u} dA_\sigma \right) \\
&\leq 2\pi(2g-2) \left(\int_S \frac{|\alpha|^2}{g_\sigma^2} e^{-2u} dA_\sigma \right).
\end{aligned}
$$

(3.2)

Relations (3.1) and (3.2) imply, for any solution (t, u), the following inequality holds

$$
2\pi(2g-2) > t^2 \frac{\left(\int_{S_\sigma} \frac{|\alpha|}{g_\sigma} \right)^2}{2\pi(2g-2)},
$$

and the conclusion follows.   □

Remark: *One can see above application of the Cauchy-Schwarz inequality as a comparison of two metrics on Teichmüller space: the Teichmüller metric and the Weil-Petersson metric (See the Proposition 2.4 of* [**McM00**]*). For the holomorphic quadratic differential αdz^2, its Teichmüller norm is $\|\alpha\|_T = \int_S |\alpha| dz d\bar{z}$, while its Weil-Petersson norm is given by $\|\alpha\|_{WP} = \sqrt{\int_S \frac{|\alpha|^2}{g_\sigma^2} dA_\sigma}$.*

We want to point out that if two solutions to (1.2) do exist, then their geometric properties are different, i.e.,

Theorem 3.2. *Assume the Gauss equation (1.2) admits two solutions $u_1 \not\equiv u_2$, and consider their associated principal curvature $\lambda_i(x) = \frac{|\alpha|}{g_\sigma} e^{-2u_i}$. Then $\lambda_1 \not\equiv \lambda_2$.*

Proof. Assume $\lambda_1 \equiv \lambda_2$. Setting the intrinsic curvature $K(x) := -(1 + \lambda_1^2(x))$, we see that the functions u_i solves the equation

$$
-\Delta u_i - 1 = K(x) e^{2u_i} (i = 1, 2).
$$

Hence we have

(3.3) $- \Delta(u_2 - u_1) = K(x)\left(e^{2u_2} - e^{2u_1} \right),$

and therefore

(3.4) $\int_S |\nabla(u_1 - u_2)|^2 = \int_S K(x)\left(e^{2u_2} - e^{2u_1} \right)(u_2 - u_1).$

Since $(e^s - e^t)(s - t) \geq 0$ for any $s, t \in \mathbb{R}$, while $K(x) \leq -1 < 0$, equality (3.4) implies that $u_2 - u_1 \equiv C$ for some constant $C \in \mathbb{R}$. Clearly $K \not\equiv 0$, and therefore equality (3.3) implies $u_2 \equiv u_1$.   □

Since the solutions in above theorem are in the same conformal structure, the Theorem 3.2 can also be seen as a consequence of a comparison theorem for conformal metrics of negative curvature ([**Wol82**]).

3.2. **Uhlenbeck's solution curve.** A solution curve to the Gauss equation can be obtained from the implicit function theorem, as in [**Uhl83**]. In this subsection, we study this solution curve further, in anticipation of using it to construct our mountain pass solution.

Consider the nonlinear map $F : W^{2,2}(S) \times [0, \infty) \to L^2(S)$ defined by

$$(3.5) \qquad F(u, t) = \Delta u + 1 - e^{2u} - \frac{t^2 |\alpha|^2}{g_\sigma^2} e^{-2u},$$

where $W^{2,k}(S)$ stands for the classical Sobolev space. At each $t \geq 0$ fixed, the linearized operator $L(u, t) : W^{2,2}(S) \to L^2(S)$ associated to F is given by

$$(3.6) \qquad L(u, t) = -\Delta + 2 \left(e^{2u} - \frac{t^2 |\alpha|^2}{g_\sigma^2} e^{-2u} \right),$$

and the differential (Fréchet derivative) of F is given by

$$(3.7) \qquad dF(u, t)(\dot{u}, \dot{t}) = -L\dot{u} - 2t\dot{t} \frac{|\alpha|^2}{g_\sigma^2} e^{-2u},$$

where $(\dot{u}, \dot{t}) \in W^{2,2}(S) \times \mathbb{R}$.

The linear operator L is geometrically meaningful, since its eigenvalues are closely related to the stability of the minimal immersion by the following theorem of Uhlenbeck:

Theorem 3.3. [**Uhl83**] *A minimal immersion with data $(\sigma, t\alpha)$ in any hyperbolic three-manifold M^3 is stable if and only if $L \geq 0$.*

From the analytic point of view, the linear operators $L(u, t)$ in (3.6) and $dF(u, t)$ in (3.7) are important in order to apply the implicit function theorem. When the linearized operator L has all positive eigenvalues, the the differential operator dF is onto. When zero is the lowest eigenvalue for L, its kernel and cokernel are one-dimensional, and dF is still onto.

We easily see that there exists a constant $\tau := \tau(\sigma, \alpha)$, such that at any solution $F(u, t) = 0$ with $t > \tau$, the first eigenvalue of $L(u, t)$ is negative. Indeed since any solution satisfies $u < 0$ (as in Theorem 3.1), from

$$e^{2u(t)} - \frac{t^2 |\alpha|^2}{g_\sigma^2} e^{-2u(t)} < 1 - \frac{t^2 |\alpha|^2}{g_\sigma^2},$$

we readily see that for large t the first eigenvalue of $L(u, t)$ is negative at any possible solution. On the other hand, we have $F(0, 0) = 0$ and we immediately see that $L(0, 0) > 0$. Hence by applying the implicit function theorem, starting from this trivial solution, one obtains a smooth solution curve γ. More specifically we obtain

Theorem 3.4. [**Uhl83**] *There exists a smooth curve*

$$\gamma : [0, \tau_0] \to W^{2,2}(S) \times [0, \infty) \qquad t \mapsto (u(t), t),$$

such that

(a) $\gamma(0) = (0, 0)$ *and* $F(\gamma(t)) = 0$ *for all* $t \in [0, \tau_0]$,

    (b) $L(u(t), t) > 0$ *for all* $t \in [0, \tau_0)$,

    (c) $\mathrm{Ker}\big(L(u(\tau_0), \tau_0)\big) \neq \{0\}$.

Note that by standard regularity theory, the solution $u(t)$ obtained in the above theorem belongs to $C^\infty(S)$.

3.3. **Variational setting.** In the next three subsections, we prove our main theorems. Since we focus on finding additional solutions, we will not use the parametrization of t in these subsections.

In this subsection, we develop the variational setting of the problem. We will introduce a different but equivalent norm to make use of the mountain pass theorem.

Setting $V(z) = \frac{t^2 |\alpha|^2}{g_\sigma^2} \geq 0$, the Gauss equation (1.1) is given by:

$$\Delta u + 1 - e^{2u} - V(z)e^{-2u} = 0, \quad \text{on } (S, \sigma),$$

which is the Euler-Lagrange equation of

$$(3.8) \qquad I(u) := \frac{1}{2}\int_S |\nabla u|^2 - \int_S \left(u - \frac{e^{2u}}{2} \right) - \int_S V(z)\frac{e^{-2u}}{2}, \quad u \in H^1(S).$$

To derive compactness property, we introduce an equivalent norm on the Sobolev space and consider a new functional whose set of critical points coincides with the one of (3.8).

To reach this goal, by choosing $\theta > 2$, we define a function $F_1 \in C^\infty(\mathbb{R})$ which satisfies

$$F_1(s) := \begin{cases} s - \frac{1}{2}e^{2s} & \text{if } s \leq 0 \\ -s^\theta & \text{if } s > 1 \end{cases} \qquad F_1'(s) < 0 \quad \forall s > 0,$$

and $F_2 \in C^\infty(\mathbb{R})$ defined as

$$F_2(s) := \begin{cases} \frac{1}{2}(s^2 + e^{-2s}) & \text{if } s \leq 0 \\ 0 & \text{if } s > 1 \end{cases} \qquad F_2' \leq 0.$$

Setting $f_i(s) := F_i'(s)$ $(i = 1, 2)$, we explicitly have

$$f_1(s) := \begin{cases} 1 - e^{2s} & \text{if } s \leq 0 \\ -\theta s^{\theta-1} & \text{if } s > 1 \end{cases} \qquad f_2(s) := \begin{cases} s - e^{-2s} & \text{if } s \leq 0 \\ 0 & \text{if } s > 1 \end{cases},$$

which coincide for $s \leq 0$ with the nonlinearities arising in the Gauss equation (1.1), and have the following property:

$$(3.9) \qquad\qquad f_1(s) < 0 \quad \forall s > 0, \qquad f_2(s) \leq 0 \quad \forall s \in \mathbb{R}.$$

Proposition 3.5. *The Gauss equation* (1.1) *is equivalent to the new equation*

$$(3.10) \qquad\qquad -\Delta u + V(z)u - \big(f_1(u) + V(z)f_2(u)\big) = 0.$$

Proof. The maximum principle easily implies that $u < 0$. From the explicit formulas of $f_1(u)$ and $f_2(u)$ for $u < 0$, it is easy to verify that

$$\begin{aligned} -\Delta u + V(z)u - f_1(u) - V(z)f_2(u) &= -\Delta u - 1 + e^{2u} + V(z)e^{-2u} \\ &= 0. \end{aligned}$$

Now we apply the maximum principle on (3.10), and make use of the properties (3.9), we easily see that the solutions of (3.10) are negative. Hence the sets of solutions of (1.1) and (3.10) coincide. $\qquad\square$

We will work with (3.10), since it admits a variational formulation that satisfies compactness property. More specifically, we consider the usual Sobolev space

$$H^1(S) := \{u \in L^2(S) : \nabla u \in L^2(S)\},$$

endowed with the following inner products

$$\langle f, g \rangle := \int_S \left\{ \nabla f \nabla g + fg \right\}. \qquad \langle f, g \rangle_V := \int_S \left\{ \nabla f \nabla g + V(z) fg \right\},$$

and denote $\| \cdot \|_{H^1}$, $\| \cdot \|_V$ as their associated norms, respectively.

Lemma 3.6. *The norms $\| \cdot \|_{H^1}$, $\| \cdot \|_V$ are equivalent.*

Proof. Since $V \in L^\infty(S)$, we clearly have $\| \cdot \|_V \leq C \| \cdot \|_{H^1}$.

Setting $\bar{u} := \frac{1}{|S|} \int_S u$, where $|S|$ is the hyperbolic area of the surface, we have

$$(3.11) \qquad \qquad \|u - \bar{u}\|_{L^2} \leq C \|\nabla u\|_{L^2}$$

from the Poincaré inequality. Furthermore,

$$
\begin{aligned}
\int_S V(z) |\bar{u}|^2 &\leq 2 \int_S \left\{ V(z) |\bar{u} - u|^2 + V(z) u^2 \right\} \\
&\leq C \int_S \left\{ |\bar{u} - u|^2 + V(z) u^2 \right\} \\
&\leq C \int_S \left\{ |\nabla u|^2 + V(z) u^2 \right\}.
\end{aligned}
$$

Therefore, since $\int_S V > 0$ we get

$$(3.12) \qquad |\bar{u}|^2 \leq C \left(\int_S V(z) \right)^{-1} \int_S \left\{ |\nabla u|^2 + V(z) u^2 \right\} \leq C \|u\|_V.$$

Using then (3.11) and (3.12), we conclude

$$\|u\|_{L^2} \leq \|u - \bar{u}\|_{L^2} + \|\bar{u}\|_{L^2} \leq C \|u\|_V.$$

This immediately implies $\|u\|_{H^1} \leq C \|u\|_V$, which completes the proof. $\qquad\square$

Now we can define the associated functional for the equation (3.10). In the Hilbert space $H^1(S)$, the functional

$$(3.13) \quad \mathcal{F}(u) := \frac{1}{2} \int_S \left\{ |\nabla u|^2 + V(z) u^2 \right\} - \int_S \left\{ F_1(u) + V(z) F_2(u) \right\}, \quad u \in H^1(S),$$

is by the Moser-Trudinger inequality well defined, of class C^1, and its critical points are weak solutions of (3.10). In this functional setting we will be able to use a minimax argument to derive a second solution to the Gauss equation.

3.4. **Mountain pass structure.** In this subsection, we show that the functional $\mathcal{F}$ exhibits a mountain pass geometry. We need the following compactness property,

Theorem 3.7. *The functional $\mathcal{F}$ satisfies the Palais-Smale condition, i.e., any sequence $\{u_n\}$ in $H^1(S)$ satisfying,*

$$(3.14) \qquad |\mathcal{F}(u_n)| \le C, \qquad \|\mathcal{F}'(u_n)\|_{H^{-1}} \to 0,$$

admits a subsequence converging strongly in $H^1(S)$.

Proof. Consider the exponent $\theta > 2$ appearing in the definition F_1. We claim that

$$F_1(s) \le \frac{s}{\theta} f_1(s) + O(1), \qquad F_2(s) \le \frac{s}{\theta} f_2(s) + O(1).$$

Indeed from our definition of F_i $(i = 1, 2)$, note first that for all $s \in (1, \infty)$

$$F_1(s) = \frac{1}{\theta} s f_1(s), \qquad F_2(s) = s f_2(s) = 0.$$

Secondly, for $s \in [0, 1]$ we obviously have:

$$F_i(s) \le s f_i(s) + O(1) \quad (i = 1, 2).$$

Thirdly, for $s < 0$ we note that

$$F_1(s) \le s \le \frac{s}{\theta} = \frac{s}{\theta}(1 - e^{2s}) + O(1) = \frac{s}{\theta} f_1(s) + O(1),$$

$$F_2(s) \le e^{-2s} \le \frac{1}{\theta}(-s)e^{-2s} + O(1) \le \frac{s}{\theta} f_2(s) + O(1).$$

Hence, we have:

$$(3.15) \qquad F_1(s) + V(z)F_2(s) \le \frac{s}{\theta}\Big(f_1(s) + V(z)f_2(s)\Big) + O(1).$$

To proceed with the proof, given a sequence $\{u_n\}$ satisfying (3.14), we prove that it is bounded. Condition (3.14) implies that

$$(3.16) \qquad \frac{1}{2}\|u_n\|_V^2 - \int_S \Big\{F_1(u_n) + V(z)F_2(u_n)\Big\} = O(1)$$

$$(3.17) \qquad \|u_n\|_V^2 - \int_S u_n\Big\{f_1(u_n) + V(z)f_2(u_n)\Big\} = o(1).$$

Using successively (3.16), (3.15) and (3.17), we deduce

$$\begin{aligned}
\frac{1}{2}\|u_n\|_V^2 &\le C + \int_S \Big\{F_1(u_n) + V(z)F_2(u_n)\Big\} \\
&\le C + \frac{1}{\theta}\int_S u_n\Big\{f_1(u_n) + V(z)f_2(u_n)\Big\} \\
&= O(1) + \frac{1}{\theta}\|u_n\|_V^2.
\end{aligned}$$

Therefore for some $\theta > 2$, we have

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_V^2 = O(1).$$

So $\|u_n\|_V = O(1)$, and therefore we have $u_n \rightharpoonup \hat{u}$ weakly in $H^1(S)$ (up to a subsequence).

Furthermore, we have

$$(3.18) \qquad \|u_n - \hat{u}\|_V^2 = \int_S \left\{ f_1(u_n)(u_n - \hat{u}) + V(z)f_2(u_n)(u_n - \hat{u}) \right\} + o(1).$$

By Lebesgue dominated convergence, we readily have $f_1(u_n) \to f_1(\hat{u})$ strongly in $L^2(S)$. Moreover, using the Moser-Trudinger inequality, it is known that the map

$$H^1(S) \to L^2 \quad u \mapsto e^u$$

is compact. Therefore we also have $f_2(u_n) \to f_2(\hat{u})$ strongly in $L^2(S)$. Hence (3.18) implies that $\|u_n - \hat{u}\|_V = o(1)$. By Lemma 3.6, we conclude that $\|u_n - \hat{u}\|_{H^1} = o(1)$. $\qquad \square$

We now show that the functional $\mathcal{F}$ admits mountain pass type solutions

Proposition 3.8. *For each $t \in (0, \tau_0)$ where τ_0 is defined in Theorem 3.4, the functional $\mathcal{F}$ admits at least two solutions.*

Proof. Using the definition F_i $(i = 1, 2)$, we see that at each point $u \in H^1(S)$ with $u \leq 0$ we have $I(u) = \mathcal{F}(u)$ and the second derivative at u satisfy

$$(3.19) \qquad I''_{(u)}(\xi, \xi) = \mathcal{F}''_{(u)}(\xi, \xi) \quad \forall \xi \in H^1(S),$$

and this bilinear form is explicitly given by

$$I''_{(u)}(\xi, \xi) = \int_S |\nabla \xi|^2 + 2 \int_S \left(e^{2u} - t^2 \frac{|\alpha(x)|^2}{g_\sigma^2} e^{-2u} \right) \xi^2.$$

Now at the stable solution $u(t)$ obtained by Uhlenbeck's Theorem 3.4, we note that for $t \in (0, \tau_0)$, standard results show that the first eigenvalue of the linearized operator $L(u(t), t)$ is given by the infimum of the Rayleigh quotient $\frac{I''_{(u(t))}(\xi, \xi)}{\|\xi\|_{H^1}^2}$ with $\xi \in H^1(S) \setminus \{0\}$. Therefore for each $t \in (0, \tau_0)$, we deduce

$$(3.20) \qquad I''_{(u(t))}(\xi, \xi) \geq C\|\xi\|_{H^1}^2 \quad \forall \xi \in H^1(S),$$

i.e. $u(t)$ is a local minimizer of the functional I. Since $u(t) < 0$, equality (3.19), with (3.20) and Lemma 3.6 imply

$$\mathcal{F}_{(u(t))}(\xi, \xi) \geq \|\xi\|_V.$$

Hence there exists a ball $B(u(t), r)$ in the Hilbert space $H^1(S)$ such that

$$(3.21) \qquad \inf_{u \in \partial B(u(t), r)} \mathcal{F}(u) \geq \mathcal{F}(u(t)).$$

Furthermore, take w to be a constant negative function, we easily see that

$$\mathcal{F}(w) = -\left(w - \frac{1}{2} e^{2w} \right)|S| - e^{-2w} \int_S V(z),$$

namely

$$\lim_{w \to -\infty} \mathcal{F}(w) = -\infty.$$

Hence, there exists $w \in H^1(S)$ such that

$$(3.22) \qquad w \notin B(u(t), r), \qquad \mathcal{F}(w) < \mathcal{F}(u(t)).$$

Since the Palais-Smale condition is satisfied, conditions (3.21) and (3.22) allow to apply the Mountain-Pass Theorem of Ambrosetti-Rabinowitz [**AR73**].   □

Proof of Theorem 1.2: Now the theorem follows from the Proposition 3.8 and the Proposition 3.5.   □

3.5. The asymptotic geometry.
We show now that as $t \to 0$ the mountain pass solutions blow-up.

Proposition 3.9. *Let (t_n, u_n) be a sequence of critical points with $t_n \to 0$. Then, along a subsequence, the following alternative holds*

  (i) $\|u_n\|_{H^1} \to 0$;
  (ii) *or* $\|u_n\|_{H^1} + \|u_n\|_\infty \to \infty$.

Proof. We have two possibilities: either $\|u_n\|_{H^1} = O(1)$ or $\|u_n\|_{H^1} \to \infty$ (up to a subsequence).

Case 1: Assume $\|u_n\|_{H^1} = O(1)$.
Then $u_n \rightharpoonup \bar{u}$, and by the Moser-Trudinger inequality we know that $e^{\pm u_n} \to e^{\pm \bar{u}}$ in $L^2(S)$. Since for each $\xi \in C^\infty(S)$ we have

$$\int_S \nabla u_n \nabla \xi = \int_S \xi - \int_S \{e^{u_n} + t_n^2 |\alpha(x)|^2 e^{-u_n}\}\xi,$$

for $n \to \infty$ we get

$$\int_S \nabla \bar{u} \nabla \xi = \int_S \xi - \int_S e^{\bar{u}}\xi$$

Therefore $-\Delta \bar{u} = 1 - e^{\bar{u}}$, which implies $\bar{u} = 0$.

Case 2: Assume $\|u_n\|_{H^1(S)} \to \infty$.
From the identity

$$(3.23) \qquad \int_S |\nabla u_n|^2 = \int_S u_n - \int_S \{e^{u_n} + t_n^2 \frac{|\alpha|^2}{|g_\sigma|^2} e^{-u_n}\}u_n,$$

we immediately see that $\|u_n\|_\infty \to \infty$. Indeed, if this is not the case we would have $\|u_n\|_L^2 = O(1)$ and (3.23) would imply $\int_S |\nabla u_n|^2 \le C$.   □

Proof of Theorem 1.3: Recall that the positive principal curvature of the minimal immersion given by $u_n(t_n)$ is given by

$$\lambda(t_n) = \frac{t_n e^{2u_n}|\alpha|}{g_\sigma}.$$

We use this to examine our options from the above Proposition (3.9). In the case one, the principal curvatures are small for t near zero. Since in a neighborhood of $(0,0)$, the functional admits a unique branch of solution (t, u_t), we conclude that u_n coincides with the solution u_t obtained by the implicit function theorem from Uhlenbeck's theorem. Since the principal curvatures are small (less than one in absolute value), the normal bundle is an almost Fuchsian three-manifold.

While in the case two, we find $|\lambda(t_n)| \to \infty$ as $t_n \to 0$. Indeed assume on the contrary that $\lambda(t_n)$ stays uniformly bounded as $t_n \to 0$. Let x_n be such that $\min u = u_{t_n}(x_n) \to -\infty$. By the minimum principle we deduce that

$$1 - (1 + \lambda(t_n)^2)e^{2u_{t_n}(x_n)} \leq 0.$$

If $\lambda(t_n)$ stays uniformly bounded we get a contradiction. This proves Theorem 1.3. $\qquad\square$

References

[AR73]   A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[CM04a] Tobias H. Colding and William P. Minicozzi, II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates off the axis for disks*, Ann. of Math. (2) **160** (2004), no. 1, 27–68.

[CM04b] ———, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks*, Ann. of Math. (2) **160** (2004), no. 1, 69–92.

[CM04c] ———, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains*, Ann. of Math. (2) **160** (2004), no. 2, 523–572.

[CM04d] ———, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected*, Ann. of Math. (2) **160** (2004), no. 2, 573–615.

[FHS83] Michael Freedman, Joel Hass, and Peter Scott, *Least area incompressible surfaces in 3-manifolds*, Invent. Math. **71** (1983), no. 3, 609–642.

[GHW10] Ren Guo, Zheng Huang, and Biao Wang, *Quasi-fuchsian three-manifolds and metrics on Teichmüller space*, Asian J. Math. (2010), to appear, arXiv:0909.2426v2.

[GT83]   David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.

[Has05] Joel Hass, *Minimal surfaces and the topology of three-manifolds*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 705–724.

[Hop89] Heinz Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1989.

[HW09]   Zheng Huang and Biao Wang, *Mean curvature flows in almost fuchsian manifolds*, preprint, arXiv:1001.4217 (2009).

[KS07]   Kirill Krasnov and Jean-Marc Schlenker, *Minimal surfaces and particles in 3-manifolds*, Geom. Dedicata **126** (2007), 187–254.

[KS08]   ———, *On the renormalized volume of hyperbolic 3-manifolds*, Com. Math. Phy. **279** (2008), no. 3, 637–668.

[KW74]   Jerry L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), 14–47.

[KW75]   ———, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. of Math. (2) **101** (1975), 317–331.

[LJ70]   H.B. Lawson Jr., *Complete minimal surfaces in S^3*, Ann. of Math. **92** (1970), 335–374.

[McM00] Curtis T. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, Ann. of Math. (2) **151** (2000), no. 1, 327–357.

[MIY82] William H. Meeks III and Shing-Tung Yau, *The classical Plateau problem and the topology of three-dimensional manifolds*, Topology **21** (1982), no. 4, 409–442.

[Rub05] J. Hyam Rubinstein, *Minimal surfaces in geometric 3-manifolds*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 725–746.

[Str00]   Michael Struwe, *Variational methods*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 34, Springer-Verlag, Berlin, 2000, Applications to nonlinear partial differential equations and Hamiltonian systems.

[SU82]   J. Sacks and K. Uhlenbeck, *Minimal immersions of closed Riemann surfaces*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 639–652.

[SY79]   Richard Schoen and Shing-Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142.

[Tau04]   Clifford Henry Taubes, *Minimal surfaces in germs of hyperbolic 3-manifolds*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 69–100 (electronic).

[Uhl83]   Karen K. Uhlenbeck, *Closed minimal surfaces in hyperbolic 3-manifolds*, Seminar on minimal submanifolds, Ann. of Math. Stud., vol. 103, Princeton Univ. Press, Princeton, NJ, 1983, pp. 147–168.

[Wol82]   Scott A. Wolpert, *A generalization of the Ahlfors-Schwarz lemma*, Proc. Amer. Math. Soc. **84** (1982), no. 3, 377–378.

DEPARTMENT OF MATHEMATICS, THE CITY UNIVERSITY OF NEW YORK, STATEN ISLAND, NY 10314, USA.

   E-mail address: `zheng.huang@csi.cuny.edu`

DEPARTMENT OF MATHEMATICS, THE CITY UNIVERSITY OF NEW YORK, STATEN ISLAND, NY 10314, USA.

   E-mail address: `mlucia@math.csi.cuny.edu`